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Curtain, Ruth; Rodman, Leiba

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ANALYTIC SOLUTIONS OF MATRIX RICCATI EQUATIONS WITH ANALYTIC COEFFICIENTS*

RUTH CURTAIN† AND LEIBA RODMAN‡

Abstract. For matrix Riccati equations of platoon-type systems and of systems arising from PDEs, assuming the coefficients are analytic or rational functions in a suitable domain, analyticity of the stabilizing solution is proved under various hypotheses. General results on analytic behavior of stabilizing solutions are developed as well.

Key words. analytic invariant subspaces, Hamiltonian, Riccati equations, spatially invariant systems, linear quadratic regulator, H-infinity control

AMS subject classifications. 47A15, 49N10, 93C05, 93C20, 93D15

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1. Introduction. The basic object studied in this paper is a $2n \times 2n$ (complex) matrix, $H \in \mathbb{C}^{2n\times 2n}$. An $n \times n$ matrix P is called *H-stabilizing* if the subspace $G(P) := \mathrm{Range} \left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right]$ of $\mathbb{C}^{2n}$ is A-invariant and all eigenvalues of the restriction of A to $G(P)$ have negative real parts.

We say that a $p\times q$ matrix $A(z)$ whose entries are functions of the complex variable z (in short, $A(z)$ is a *matrix function*) is *analytic*, respectively, *rational*, *continuous*, etc., if every entry of $A(z)$ is an analytic, respectively, rational, continuous, etc., function of z.

In control systems, the $2n \times 2n$ matrix H often represents the Hamiltonian of a system. Upon partition

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad H_{ij} \in \mathbb{C}^{n\times n},$$

the H-invariance of $G(P)$ takes the form of a *Riccati equation*

$$-H_{22}P + PH_{11} + PH_{12}P - H_{21} = 0, \tag{1.1}$$

and the restriction of H to $G(P)$ is similar to $H_{11} + H_{12}P$. In the case that $H(z)$ is an analytic or continuous matrix function, it is of considerable interest, both from theoretical and applied points of view, to find out whether or not an $H(z)$-stabilizing matrix (provided it exists) can also be chosen to be analytic or continuous. This problem has been addressed, mainly in the control systems literature, see [25, 11], where a proof of real analyticity is given (under suitable hypotheses and symmetries), [24, 17, 8, 9], and references there for the continuity properties (as function of the coefficients), again under suitable hypotheses that involve symmetries. General results on smoothness properties of unmixed solutions are given in [1, Chapter 4.2].

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†Department of Mathematics and Computing Science, University of Groningen, 97400 AV Groningen, The Netherlands (r.f.curtain@rug.nl).
‡Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795 (lxrodm@math.wm.edu). This author's research was supported by the Faculty Research Assignment and Plumeri Award at the College of William and Mary.

More recently, there has been renewed interest in the analyticity property due to applications in the control of spatially invariant systems (see [2, 22]), where one seeks spatially decaying feedbacks. In the case of the linear quadratic regulator problem, the key to establishing this property is to show that the stabilizing solution to the Riccati equation has an analytic extension. A result in this direction was shown in [2, Theorem 6] (see section 6). While the spatial decaying property for spatially invariant systems is the main motivation for our study, there are numerous other applications, for example, in H-infinity control design. Using the results and techniques from the theory of analytic invariant subspaces in [13], we analyze in detail the analytic properties of stabilizing matrices for Hamiltonian matrices H whose entries are analytic functions in a connected open set Ω in the complex plane. We illustrate their potential for control applications with two classes of spatially invariant systems. The analytic results in [2] follow as particular cases.

In section 2 we prove a general result concerning conditions for the existence of a meromorphic solution $P(z)$ to the Riccati equation for $z \in \Omega$. Moreover, we show that in the particular case that the entries of $H(z)$ are rational functions, $P(z)$ has at most finitely many poles in Ω. Our main result in section 3 is that if for each $z \in \Omega$ there exists a unique $H(z)$-stabilizing solution, then $P(z)$ is analytic for $z \in \Omega$. Sections 4–7 concern applications of these general results to the linear quadratic regulator problem for two types of spatially invariant systems. First the platoon-type systems are introduced in section 4, and the desirability of the spatially decaying property and its connection to analyticity are explained. This translates into finding conditions for solutions to a Riccati equation to be analytic in an annulus around the unit circle, which is done in section 5. Sections 6 and 7 treat the analogous problems for partial differential systems defined on an infinite spatial domain. In this case the relevant problem is to find conditions for solutions to a Riccati equation to be analytic in a vertical strip around the imaginary axis, and this is done in section 7.

2. Analyticity of stabilizing matrices.

In this and the next section we develop general results concerning analyticity of stabilizing matrices under the assumption that H is an analytic matrix function.

THEOREM 2.1. *Let $H(z)$ be an analytic $2n \times 2n$ matrix function of the complex variable z on a connected open set Ω in the complex plane. If the following two assumptions hold:*

(a) *$H(z)$ has no eigenvalues on the imaginary axis, for every $z \in \Omega$;*

(b) *For some $z_0 \in \Omega$, $H(z_0)$ is similar to either $-H(z_0)$ or $-H(z_0)^*$;*

then either (A) *or* (B) *holds:*

(A) *There are no $H(z)$-stabilizing matrices for any $z \in \Omega$;*

(B) *There is a meromorphic on Ω $n \times n$ matrix function $P(z)$ such that $P(z_0)$ is the unique $H(z_0)$-stabilizing matrix if z_0 is not a pole of $P(z)$, and there are no $H(z_0)$-stabilizing matrices if z_0 is a pole of $P(z)$.*

Proof. In view of condition (b), if λ is an eigenvalue of $H(z_0)$, then $-\lambda$ (or $-\overline{\lambda}$) is an eigenvalue as well and with the same partial multiplicities as λ. Taking into account (a), we see that the spectral subspace of $H(z_0)$ corresponding to the eigenvalues in the open left (respectively, open right) half plane has dimension n. In view of (a), this property holds for every $H(z)$, $z \in \Omega$. So if $P(z')$ is an $H(z')$-stabilizing matrix for $H(z')$ for some $z' \in \Omega$, then $G(P(z'))$ must be the spectral subspace of $H(z')$ corresponding to the eigenvalues in the open left half plane. In particular, an $H(z)$-stabilizing matrix is unique (if it exists) for every $z \in \Omega$. Now assuming (A) does not hold, (B) follows from the analyticity of the spectral $H(z)$-

invariant subspace corresponding to the eigenvalues of $H(z)$ in the open left half plane (see [13, Theorem 18.7.2]); the poles of $P(z)$ are exactly the points at which this subspace is not a graph subspace. ∎

In general, the region of analyticity of P will be larger than the region where $P(z)$ is $H(z)$-stabilizing, as the following two examples illustrate.

Example 2.2. Consider the analytic function

$$H(z) = \begin{bmatrix} 1+z & -1+2z^2 \\ -1 & -1+z \end{bmatrix},$$

noting that assumption (b) of Theorem 2.1 is satisfied in $z = 0$. For condition (a), let us examine where $H(z)$ has imaginary eigenvalues. This reduces to finding a real ω_0 and points $z \in \mathbb{C}$ such that

$$\jmath\omega_0 = z \pm \sqrt{2 - 2z^2}.$$

In fact this has infinitely many solutions parametrized by ω_0, namely,

$$z = \left(\pm\sqrt{6 + 2\omega_0^2} + \jmath\omega_0\right)/3.$$

These points lie on the curve $\{x + \jmath y \,:\, x^2 - 2y^2 = 2/3\}$. So Theorem 2.1 predicts that, with the possible exception of poles of P, there will be $H(z)$-stabilizing matrices in the open set $\Omega := \{x + \jmath y \,:\, x^2 - 2y^2 < 2/3\}$.

Solving the associated Riccati equation gives the two possible solutions

$$P(z) = \frac{1 \pm \sqrt{2 - 2z^2}}{1 - 2z^2},$$

which clearly have algebraic branch points at $z = \pm 1$ and poles at $z = \pm 1/\sqrt{2}$. The $H(z)$-stabilizing solution is the one such that

$$A_P(z) := H_{11}(z) + H_{12}(z)P(z) = z \mp \sqrt{2 - 2z^2}$$

is stable, i.e.,

$$P_+(z) = \frac{1 + \sqrt{2 - 2z^2}}{1 - 2z^2}, \quad A_{P_+}(z) = z - \sqrt{2 - 2z^2}.$$

To examine the region where A_{P_+} is stable, we calculate the real part of $A_{P_+}(z)$. Now with $z = x + \jmath y$, we have

$$\mathrm{Re}(A_{P_+}(z)) = x - \sqrt{1 + y^2 - x^2 + \sqrt{(1 + y^2 - x^2)^2 + 4x^2y^2}}.$$

Hence the region of stability of $A_{P_+}(z)$ is described by Ω as predicted by Theorem 2.1. Note that P_+ has poles at the points $z = \pm 1/\sqrt{2}$, and so at these points there is no H-stabilizing matrix. Outside Ω there will be no $H(z)$-stabilizing solutions, but $P(z)$ will be analytic, except at the points $z = \pm 1$.

Example 2.3. Consider the analytic function

$$H(z) = \begin{bmatrix} -1+z & -1+2z^2 \\ -1 & 1+z \end{bmatrix},$$

noting that assumption (b) of Theorem 2.1 is satisfied in $z = 0$. As in Example 2.2, Theorem 2.1 predicts that there will be $H(z)$-stabilizing matrices in the open set Ω of Example 2.2. Solving the associated Riccati equation gives the two possible solutions

$$P(z) = \frac{-1 \pm \sqrt{2 - 2z^2}}{1 - 2z^2},$$

which clearly have algebraic branch points at $z = \pm 1$ and poles at $z = \pm 1/\sqrt{2}$. The $H(z)$-stabilizing solution is given by

$$P_+(z) = \frac{-1 + \sqrt{2 - 2z^2}}{1 - 2z^2}, \quad A_{P_+}(z) = z - \sqrt{2 - 2z^2}.$$

Note that the apparent poles at $z = \pm 1/\sqrt{2}$ cancel out. As for Example 2.2, the region of stability of $A_{P_+}(z)$ is described by Ω, and in this region, $P_+(z)$ is $H(z)$-stabilizing. Again, outside Ω, there will be no $H(z)$-stabilizing solutions, but $P(z)$ will be analytic, except at $z = \pm 1$.

The above examples are typical of the behavior when the entries of $H(z)$ are analytic and rational.

THEOREM 2.4. *Assume in addition to the hypotheses of Theorem* 2.1 *that the entries of $H(z)$ are rational functions with poles outside Ω. If* (B) *of Theorem* 2.1 *holds, then the number of poles of $P(z)$ is either empty (i.e., $P(z)$ is analytic) or finite.*

We need some preliminaries for the proof of Theorem 2.4.

Denote by R_Ω the algebra of scalar rational functions with poles outside the connected open set Ω. Fix $g_0(z), \dots, g_{n-1}(z) \in R_\Omega$. Consider the roots $\lambda_1(z), \dots, \lambda_n(z)$ of the polynomial equation

$$\lambda^n + \lambda^{n-1} g_{n-1}(z) + \cdots + \lambda g_1(z) + g_0(z) = 0. \tag{2.1}$$

Here the $\lambda_j(z)$'s are understood as multivalued analytic functions of $z \in \Omega$, except possibly for finitely many algebraic branch points $z_1, \dots, z_p$. The number of values of $\lambda_j(z)$ at each $z \in \Omega$ is finite and is bounded by an integer which is independent of z and j. Consider the algebra $M = M(g_0(z), \dots, g_{n-1}(z))$ of multivalued functions generated by $\lambda_1(z), \dots, \lambda_n(z)$ and R_Ω. We say that $z_0 \in \Omega$ is a zero of $f \in M$ if zero is one of the values of $f(z_0)$.

LEMMA 2.5. *If $f \in M$ has a zero at infinitely many points in Ω, then every $z_0 \in \Omega$ is a zero of f.*

Proof. Let us write $g_j(z) = p_j(z)/p_n(z)$ with $p_0, \dots, p_{n-1}, p_n \in \mathbb{C}[z]$ having no common factor. Then we can rewrite (2.1) as

$$q(\lambda, z) := p_n(z)\lambda^n + p_{n-1}(z)\lambda^{n-1} + \cdots + p_0(z) = 0.$$

Let $q = q_1^{r_1} \dots q_s^{r_s}$ be the decomposition of the polynomial $q(\lambda, z)$ into irreducible factors, and let X_i, $i = 1, \dots, s$, be the compact Riemann surface of the algebraic curve $q_i(\lambda, z) = 0$, completed with the points at infinity. We will need some elementary facts about compact Riemann surfaces which can be found in many textbooks, for example, [26, 23].

Since both z and the $\lambda_j(z)$'s are meromorphic functions on X_i for each i, the function f yields (single valued) meromorphic functions $f_1, \dots, f_s$ on $X_1, \dots, X_s$, respectively; each one of these is a multivalued (algebraic) function on Ω, and the values of f at $z \in \Omega$ are the union of the values of f_i at z, $i = 1, \ldots, s$. If f has infinitely many zeroes in Ω, then at least one of f_i has infinitely many zeroes on X_i and hence is identically zero. (Here the fact is used that the Riemann surface of an irreducible algebraic curve is connected and the obvious fact that a meromorphic function on a compact Riemann surface can have only finitely many zeros without being identically zero.) Thus, f has a zero at every $z \in \Omega$. $\square$

Proof of Theorem 2.4. The proof of Theorem 2.1 shows that

$$\text{Range} \begin{bmatrix} I \\ P(z) \end{bmatrix} = X(z), \quad z \in \Omega \setminus (\text{poles of } P(z)), \tag{2.2}$$

where $X(z)$, $z \in \Omega$, is the spectral invariant subspace of $H(z)$ corresponding to the eigenvalues in the open left half plane. In turn,

$$X(z) = \text{Range} \left(\frac{1}{2\pi i} \int_\Gamma (\lambda I - H(z))^{-1} d\lambda \right),$$

where Γ is a suitable contour that encloses all eigenvalues of $H(z)$ in the open left half plane and does not enclose any other eigenvalue of $H(z)$.

Write

$$x(\lambda, z) := \det (\lambda I - H(z)) p_{2n}(z) = p_{2n}(z)\lambda^{2n} + p_{2n-1}(z)\lambda^{2n-1} + \cdots + p_1(z)\lambda + p_0(z), \tag{2.3}$$

where the $p_j(z)$'s are polynomials of z and $p_{2n}(z)$ has no zeros in Ω. The number $r(z)$ of distinct roots $\lambda_1(z), \ldots, \lambda_{r(z)}(z)$ of the polynomial $x(\lambda, z)$ may vary with $z \in \Omega$. However,

$$r := \max\{r(z) \, : \, z \in \Omega\} = r(z) \quad \text{ for all } \quad z \in \Omega \setminus \Omega_0, \tag{2.4}$$

where Ω_0 is a certain finite (or possibly empty) subset of Ω. Indeed, (2.4) follows easily from the formula

$$r(z) = \operatorname{rank} S \left(x(\lambda, z), \frac{\partial x(\lambda, z)}{\partial \lambda} \right) - 2n + 1, \tag{2.5}$$

where $S(p(\lambda), q(\lambda))$ is the Sylvester resultant (degree p + degree q) $\times$ (degree p + degree q) matrix of two polynomials $p(\lambda)$, $q(\lambda)$. (Note that the entries of the Sylvester resultant matrix in (2.5) are polynomials of z.) See, e.g., [3, 12]. Formula (2.5) follows at once from the classical property of $S(p(\lambda), q(\lambda))$, namely, that the rank deficiency of $S(p(\lambda), q(\lambda))$ coincides with the degree of the greatest common divisor of $p(\lambda)$ and $q(\lambda)$.

For every $z \in \Omega \setminus \Omega_0$, write

$$x(\lambda, z) = p_{2n}(z) \prod_{j=1}^{r} (\lambda - \lambda_j(z))^{m_j(z)},$$

where $m_j(z)$ is the multiplicity of $\lambda_j(z)$ as a root of $x(\lambda, z)$. (We enumerate $\lambda_1(z), \ldots,$ $\lambda_r(z)$ so that the $\lambda_j(z)$'s are analytic in $\Omega \setminus \Omega_0$, with possible algebraic branch points at Ω_0.) A priori $m_j(z)$ may depend on z, however, we claim that $m_j(z)$ are independent of $z \in \Omega \setminus \Omega_0$. Indeed, for every fixed r-tuple of positive integers $N = (n_1, \ldots, n_r)$ such that $n_1 + \cdots + n_p = 2n$, let

$$\Omega_N = \{z \in \Omega \setminus \Omega_0 \, : \, m_j(z) = n_j \quad \text{for} \quad j = 1, 2, \ldots, r\}.$$

Obviously, the sets Ω_N and $\Omega_{N'}$ are disjoint if $N \neq N'$; on the other hand, each set Ω_N is easily seen to be open in $\Omega \setminus \Omega_0$ (because in the neighborhood of every fixed z_0, for every $\lambda_j(z_0)$ there is only one root of $x(\lambda, z)$ which is close to $\lambda_j(z_0)$ and therefore must have the same multiplicity as the root $\lambda_j(z_0)$ of $x(\lambda, z_0)$ has), and the union of the sets Ω_N over all possible N's is equal to $\Omega \setminus \Omega_0$. Since the number of all possible N's is finite, we must have that only one of the sets Ω_N is non-empty, proving that $m_j := m_j(z)$ actually are independent of $z \in \Omega \setminus \Omega_0$.

Letting $\operatorname{adj} X$ be the algebraic adjoint of a matrix X, we now have for every $z \in \Omega \setminus \Omega_0$:

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma (\lambda I - H(z))^{-1} d\lambda &= \frac{1}{2\pi i}\int_\Gamma \frac{\operatorname{adj}(\lambda I - H(z))}{\det(\lambda I - H(z))} d\lambda \\ &= (p_{2n}(z))^{-1}\frac{1}{2\pi i}\int_\Gamma \frac{\operatorname{adj}(\lambda I - H(z))}{x(\lambda, z)} d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\operatorname{adj}(\lambda I - H(z))}{\prod_{j=1}^r (\lambda - \lambda_j(z))^{m_j}} d\lambda.\end{aligned}$$

Using the standard residue theorem and formulas for calculating residues (see, e.g., [19, section 4.1]) we see that the matrix $(2\pi i)^{-1}\int_\Gamma (\lambda I - H(z))^{-1} d\lambda$ has the form

$$(2\pi i)^{-1}\int_\Gamma (\lambda I - H(z))^{-1} d\lambda = (w(z))^{-1} V(z), \quad z \in \Omega \setminus \Omega_0,$$

where the scalar function $w(z)$ and the entries of the matrix $V(z)$ belong to the algebra

$$M(p_0(z)/p_{2n}(z), \ldots, p_{2n-1}/p_{2n}(z))$$

of multivalued functions; in addition, $w(z)$ does not take value zero on any point in $\Omega \setminus \Omega_0$. On the other hand, $(2\pi i)^{-1}\int_\Gamma (\lambda I - H(z))^{-1} d\lambda$ is an analytic single valued function of $z \in \Omega$ (see Theorem 18.7.2 of [13] and its proof). Therefore, $(w(z))^{-1}V(z)$ is also single valued on $\Omega \setminus \Omega_0$.

In view of (2.2), there exists $z_0 \in \Omega \setminus \Omega_0$ such that $X(z_0)$ is a graph subspace. This means that some $n \times n$ subdeterminant $D(z)$ of the $n \times (2n)$ matrix formed by the top n rows of $V(z)$ does not take a zero value (among possibly many values) at z_0; otherwise, we would obtain that all $n \times n$ subtdeterminants of the top n rows of the single valued matrix $(w(z_0))^{-1}V(z_0)$ are zeros, and hence the top n rows of $(w(z_0))^{-1}V(z_0)$ form a matrix of rank less than n, a contradiction with $X(z_0)$ being a graph subspace. By Lemma 2.5, $D(z)$ does not take a zero value for every $z \in \Omega \setminus (\Omega_0 \cup \Omega_1)$, where Ω_1 is a finite (or empty) set. For all such z, $X(z)$ is obviously a graph subspace, and since a point for which $X(z)$ is a graph subspace cannot be a pole of $P(z)$, we obtain that the poles of $P(z)$ must be contained in the finite set $\Omega_0 \cup \Omega_1$.

This completes the proof of Theorem 2.4. $\square$

3. Uniqueness implies analyticity. We consider the set $\mathbb{G}_n(\mathbb{C}^{2n})$ of all n-dimensional subspaces of $\mathbb{C}^{2n}$ (the Grassmanian). We use the standard metric topology in $\mathbb{G}_n(\mathbb{C}^{2n})$ which is defined by the *gap* $\Theta(\mathcal{M}, \mathcal{N})$ between two subspaces $\mathcal{M}, \mathcal{N} \in \mathbb{G}_n(\mathbb{C}^{2n})$:

$$\Theta(\mathcal{M}, \mathcal{N}) = \|P_\mathcal{M} - P_\mathcal{N}\|,$$

where P_Z stands for the orthogonal projection on the subspace Z and $\|\cdot\|$ is the operator norm. The following well-known properties of graph subspaces will be useful.

PROPOSITION 3.1.

(a) *The set of graph subspaces is open and dense in* $\mathbb{G}_n(\mathbb{C}^{2n})$ *(in the standard topology);*

(b) *The map*

$$\phi\left(\text{Range}\begin{bmatrix} I \\ P \end{bmatrix}\right) = P$$

is a homeomorphism from the set of graph subspaces onto $\mathbb{C}^{2n\times 2n}$.

For the proof of Proposition 3.1, as well as for other relevant properties of the topology of the Grassmanian, we refer to [13], among many sources. We mention in passing that the map ϕ and its inverse are also analytic, with respect to the standard structure of $\mathbb{G}_n(\mathbb{C}^{2n})$ as an analytic manifold.

THEOREM 3.2. *Let $H(z)$ be an analytic $2n \times 2n$ matrix function on a connected open set Ω in the complex plane. If for every $z \in \Omega$ there is a unique $H(z)$-stabilizing matrix $P(z)$, then $P(z)$ is analytic in Ω.*

Proof. A point $z_0 \in \Omega$ is said to be *regular* if there is an open neighborhood N of z_0 such that for every $H(z_0)$-invariant subspace S_0, there is a family of subspaces $S(z)$, $z \in N$, with the properties that $S(z_0) = S_0$, $S(z)$ is $H(z)$-invariant for every $z \in N$, and $S(z)$ is analytic in N. Points that are not regular are called *singular*. It follows from [13, Theorem 19.4.1] that the singular points form a discrete set.

Let $\widetilde{z}_0$ be regular. Then it follows from [13, Theorems 19.4.1 and 19.4.2] that there is a family of subspaces $S(z) \subset \mathbb{C}^{2n}$, $z \in \Omega$, with the following properties:

(1) $S(\widetilde{z}_0) = G(P(\widetilde{z}_0))$.

(2) $S(z)$ is $H(z)$-invariant for all $z \in \Omega$.

(3) $S(z)$ is analytic in Ω except possibly for a discrete set of algebraic branch points; $\widetilde{z}_0$ is not one of the branch points.

Writing

$$S(z) = \text{Range}\begin{bmatrix} I \\ Q(z) \end{bmatrix},$$

we obtain a meromorphic (except possibly for a discrete set of algebraic branch points) matrix function $Q(z)$ such that $Q(\widetilde{z}_0) = P(\widetilde{z}_0)$.

Clearly, $Q(z)$ is $H(z)$-stabilizing for z in a vicinity of $\widetilde{z}_0$.

We prove that $Q(z_0)$ has all eigenvalues in the open left half plane for every $z_0 \in \Omega$ not a pole of $Q(z)$. Suppose this is not the case. Then $Q(z_0)$ has an eigenvalue λ_0 on the imaginary axis $\mathbb{J}$ for some z_0, where z_0 is chosen so that for some sequence of points z_m tending to z_0, all eigenvalues of $Q(z_m)$ are in the open left half plane. Taking sufficiently small positive δ, there exists a set $\Gamma \subset \{z : |z - z_0| < \delta\}$ which consists of finitely many algebraic curves intersected with the disk $D := \{z : |z - z_0| < \delta\}$ such that for $z \in D$, the matrix $Q(z)$ has an eigenvalue on $\mathbb{J}$ precisely when $z \in \Gamma$. (In fact, Γ is the preimage of the map that sends z to the eigenvalues of $Q(z)$ in a vicinity of λ_0 when the preimage is restricted to a segment of the imaginary axis around λ_0; if there are several dictinct eigenvalues of $Q(z_0)$ on $\mathbb{J}$, then Γ is the union of all such preimages taken with respect to all distinct eigenvalues of $Q(z_0)$ on $\mathbb{J}$.) In view of our choice of z_0, there is a curve Γ_0 in Γ such that on one ("stabilizing") side of the curve Γ_0, the matrix $Q(z)$ is stable. Take a regular point $z' \in \Gamma_0$. Then $Q(z')$ is not stable. By our hypotheses, there exists a unique $H(z')$-stabilizing matrix $R(z')$ which must be different from $Q(z')$. Since z' is regular, $R(z')$ admits extension to an analytic family of matrices $R(z)$ in a sufficiently small open neighborhood N' of z' such that

$R(z)$ is $H(z)$-stabilizing for every $z \in N'$. However, for those values of $z \in N'$ that are on the "stabilizing" side of Γ_0, there is also another $H(z)$-stabilizing matrix, namely, $Q(z)$. This contradicts the uniqueness of the $H(z)$-stabilizing matrix.

Thus, $Q(z_0)$ is $H(z_0)$-stabilizing for all $z_0 \in \Omega$ not poles of $Q(z)$. $Q(z)$ cannot have branch points because otherwise by making one turn around a branch point, another $H(z)$-stabilizing matrix is obtained, a contradiction with uniqueness.

Finally we show that $Q(z)$ cannot have poles. Suppose z' is a pole of $Q(z)$. Let $R(z')$ be the unique $H(z)$-stabilizing matrix. The graph subspace $G(R(z'))$ must be an isolated point (with respect to the standard topology in $\mathbb{G}_n(\mathbb{C}^{2n})$) in the set of all $H(z')$-invariant subspaces because otherwise we would obtain many $H(z')$-stabilizing matrices produced by the invariant subspaces (which are necessarily graph subspaces by Proposition 3.1) which are close to $G(R(z'))$. We now use the fact (proved in [4, 6]; see also [5]) that every isolated $H(z')$-invariant subspace is stable in the sense of robustness; in other words, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every w with $|z - w| < \delta$, the matrix $H(w)$ has an invariant subspace $\mathcal{M}(w)$ such that

$$\Theta(\mathcal{M}(w), G(R(z'))) < \epsilon.$$

Taking ϵ sufficiently small, we see by Proposition 3.1 that $\mathcal{M}(w)$ must be a graph subspace,

$$\mathcal{M}(w) = \text{Range} \begin{bmatrix} I \\ R(w) \end{bmatrix},$$

thus giving rise to a matrix $R(w)$. By Proposition 3.1(b), $R(w)$ is $A(w)$-stabilizing and is different from $Q(w)$, for $w \neq z'$ sufficiently close to z'. Again, this is a contradiction with uniqueness. ☐

Several remarks concerning the results of Theorems 2.1, 2.4, and 3.2 are in order.

(1) The results hold for analytic $H(z) \in \mathbb{C}^{(m+n)\times(m+n)}$ and $P \in \mathbb{C}^{n\times m}$ with the same proof. We confined the statements to the case $m = n$ having in mind applications in control systems.

(2) The same results hold if one understands H-stabilizing P as all eigenvalues of the restriction of H to $G(P)$ being in the open unit disc rather than in the open left half plane. In fact, the results and proofs remain valid if H-stabilizing P means all eigenvalues of the restriction of H to $G(P)$ are in one side of a fixed piecewise continuously differentiable closed curve without self-intersections on the Riemann sphere; the imaginary line in Theorems 2.1 and 2.4 is then replaced by the curve.

(3) Essentially the same ideas can be used to prove an extension of Theorem 3.2: assume that the number k of $H(z)$-stabilizing matrices (for a fixed $z \in \Omega$) is independent of $z \in \Omega$. Then there exist k analytic matrix functions $P_1(z), \ldots, P_k(z)$ such that for every $z \in \Omega$, the matrices $P_j(z)$, $j = 1, 2, \ldots, k$, are distinct and form the set of $H(z)$-stabilizing matrices.

4. Platoon-type systems. The platoon-type systems we consider are described by

$$\dot{x}_r(t) = \sum_{l\in\mathbb{Z}} A_l x_{r-l}(t) + \sum_{l\in\mathbb{Z}} B_l u_{r-l}(t), \tag{4.1}$$

$$y_r(t) = \sum_{l\in\mathbb{Z}} C_l x_{r-l}(t) + \sum_{l\in\mathbb{Z}} D_l u_{r-l}(t), \tag{4.2}$$

where $r \in \mathbb{Z}$, the set of integers, $A_l \in \mathbb{C}^{n\times n}, B_l \in \mathbb{C}^{n\times m}, C_l \in \mathbb{C}^{p\times n}, D_l \in \mathbb{C}^{p\times m}$, and $x_r(t) \in \mathbb{C}^n$, $u_r(t) \in \mathbb{C}^m$, and $y_r(t) \in \mathbb{C}^p$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point $r \in \mathbb{Z}$. This class belongs to the class of spatially invariant systems introduced in [2], and it is a special type of infinite-dimensional system. Using the terminology and formalism of [10], we can formulate (4.1) and (4.2) as an infinite-dimensional linear system $\Sigma(A, B, C, D)$,

$$\begin{aligned} \dot{x}(t) &= (Ax)(t) + (Bu)(t), \\ y(t) &= (Cx)(t) + (Du)(t), \quad t \geq 0, \end{aligned} \tag{4.3}$$

with the state space $X = \ell_2(\mathbb{C}^n)$, the input space $U = \ell_2(\mathbb{C}^m)$, and the output space $Y = \ell_2(\mathbb{C}^p)$. Note that X, U, Y are all infinite dimensional. So

$$x(t) = (x_r(t))_{r\in\mathbb{Z}} \in \ell_2(\mathbb{C}^n),\ u(t) = (u_r(t))_{r\in\mathbb{Z}} \in \ell_2(\mathbb{C}^m),\ y(t) = (y_r(t))_{r\in\mathbb{Z}} \in \ell_2(\mathbb{C}^p),$$

and A, B, C, D are convolution operators. Denoting the signals and the convolution operators generically by $x(t)$ and T, respectively, we have

$$((Tx)(t))_r = \sum_{l\in\mathbb{Z}} T_l x_{r-l}(t) = \sum_{l\in\mathbb{Z}} T_{r-l} x_l(t).$$

Taking discrete Fourier transforms of the system equations (4.3), $\mathfrak{F} : \ell_2(\mathbb{C}^n) \to \mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$, we obtain

$$\begin{aligned} \dot{\check{x}}(t) &= \mathfrak{F}\dot{x}(t) = \check{A}\check{x}(t) + \check{B}\check{u}(t), \\ \check{y}(t) &= \mathfrak{F}y(t) = \check{C}\check{x}(t) + \check{D}\check{u}(t), \end{aligned} \tag{4.4}$$

where $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$, $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$, $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$, and $\check{D} = \mathfrak{F}D\mathfrak{F}^{-1}$ are multiplicative operators of the form $\check{A}(e^{\mathrm{i}\theta}) := \sum_{l\in\mathbb{Z}} A_l e^{-\mathrm{i}l\theta}$, etc.

Denote by $\mathbb{T}$ the unit circle in the complex plane. If $\check{A}, \check{B}, \check{C}, \check{D} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{\bullet\times\bullet})$, then they and A, B, C, D are all bounded operators ("$\bullet$" denotes the appropriate dimension). In this case the linear system $\Sigma(A, B, C, D)$ on the state space $\ell_2(\mathbb{C}^n)$ is isometrically isomorphic to the linear system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ on the state space $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ with input and output spaces $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^m)$ and $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^p)$, respectively. Their system theoretic properties are identical (see [10, Exercise 2.5]), and so it suffices to apply the standard theory from [10] to the particular class of infinite-dimensional systems. For almost all $z \in \mathbb{T}$, the system (4.4) can be written as

$$\begin{aligned} \frac{\partial}{\partial t}\check{x}(z,t) &= \check{A}(z)\check{x}(z,t) + \check{B}(z)\check{u}(z,t) \\ \check{y}(z,t) &= \check{C}(z)\check{x}(z,t) + \check{D}(z)\check{u}(z,t), \quad t \geq 0. \end{aligned} \tag{4.5}$$

The motivation for studying this special class of system stems from the interest shown in the literature for controlling infinite platoons of vehicles over the years (see [2, 7, 16, 18, 20, 21]). The models obtained for these configurations have the spatially invariant form (4.5), and $\check{A}, \check{B}, \check{C}, \check{D}$ have finitely many nonzero Fourier coefficients.

A matrix $\check{P}(z)$ is said to be *stabilizing* if $\check{P}(z)^* = \check{P}(z)$ and $\check{A}(z) - \check{B}(z)\check{B}(z)^*\check{P}(z)$ has all its eigenvalues in the open left half plane. The following result is well known.

THEOREM 4.1. *Suppose that $\check{A}(z), \check{B}(z), \check{C}(z)$ are continuous in z on $\mathbb{T}$. If $(\check{A}(z), \check{B}(z), \check{C}(z))$ is stabilizable and detectable for each $z \in \mathbb{T}$, then the following family of Riccati equations has a unique stabilizing solution $\check{P}(z)$ for each $z \in \mathbb{T}$:*

$$\check{A}(z)^*\check{P}(z) + \check{P}(z)\check{A}(z) - \check{P}(z)\check{B}(z)\check{B}(z)^*\check{P}(z) + \check{C}(z)^*\check{C}(z) = 0. \tag{4.6}$$

Moreover, $\check{P}(z)$ is positive semidefinite for each $z \in \mathbb{T}$.

The stabilizing feedback in Theorem 4.1, $\check{K} = -\check{B}^*\check{P}$, has the form

$$\check{K}(z) = \sum_{l\in\mathbb{Z}} K_l z^l, \quad z \in \mathbb{T},$$

and the control action has the form

$$u(t)_r = \sum_{l\in\mathbb{Z}} K_{r-l} x_l(t) = \sum_{l\in\mathbb{Z}} K_l x_{r-l}(t).$$

For practical implementation it is desirable that the control action depends only on the nearest neighbors $x_r, x_{r\pm1}, \ldots, x_{r\pm(r+s)}$, where s can be chosen as small as possible. This will be the case only if the Fourier coefficients k_r of K decay rapidly as $r \to \infty$. Consequently the authors of [22] sought conditions under which the solutions of Riccati equations will have a spatially decaying property. As they showed in [22, Theorem 1], it is sufficient that $\check{K}$ has an analytic extension to an annulus around the unit circle.

LEMMA 4.2. *Suppose that $K(z)$ is a matrix function which is analytic in the annulus*

$$\mathfrak{A}(\tau) = \{z \in \mathbb{C} \mid e^{-\tau} < |z| < e^{\tau}\}, \qquad \tau > 0. \tag{4.7}$$

Let $K(z) = \sum_{l\in\mathbb{Z}} K_l z^l$ be the Laurent series for $K(z)$. Then for every α, $0 \le \alpha < \tau$, there exists a positive μ such that $\|K_l\| \le \mu e^{-|l|\alpha}$.

So the Fourier coeficients of $\check{K}$ will decay exponentially fast if $\check{B}$ and the solution to the Riccati equation (4.6) have an analytic extension to an annulus around the unit circle.

5. Riccati equation for the platoon-type systems.

In this section we give conditions under which the solutions to the Riccati equation (4.6) have an analytic extension to an annulus around the unit circle. The analogue of the approach used in [2] is to seek an analytic extension $P(z)$ of the solution $\check{P}(z)$ to the Riccati equation (4.6) to an annulus $\mathfrak{A}(\alpha)$ (see (4.7)); i.e., $P(z) = \check{P}(z)$ for $z \in \mathbb{T}$. The obvious candidate is a solution $P(z)$ for $z \in \mathfrak{A}(\alpha)$ to the following nonstandard Riccati equation

$$A^{\sim}(z)P(z) + P(z)A(z) - P(z)B(z)B^{\sim}(z)P(z) + C^{\sim}(z)C(z) = 0, \tag{5.1}$$

where $A^{\sim}(z) := A(\overline{z^{-1}})^*$, and we suppose that $A(z), B(z), C(z)$ are $n \times n, n \times m$, and $p \times n$ matrix valued functions. In fact, we consider a more general equation

$$A^{\sim}(z)P(z) + P(z)A(z) - P(z)D(z)P(z) + Q(z) = 0, \tag{5.2}$$

where $D(z) = D^{\sim}(z)$ and $Q(z) = Q^{\sim}(z)$ are $n \times n$. The Hamiltonian matrix function is given by

$$H(z) = \begin{bmatrix} A(z) & -D(z) \\ -Q(z) & -A^{\sim}(z) \end{bmatrix}, \tag{5.3}$$

where $D(z) = B(z)B^{\sim}(z)$, $Q(z) = C^{\sim}(z)C(z)$. We suppose that $H(z)$ is analytic for $z \in \mathfrak{A}(\alpha)$. Note that $H(z)$ and $-H^{\sim}(z)$ are similar matrices:

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} H(z) = -H^{\sim}(z) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{5.4}$$

Remark 5.1. From the above it follows that if $\lambda(z), (x(z), y(z))^T$ is a stable eigenpair for $H(z)$, then $-\lambda(z), (y(z), -x(z))^T$ is an unstable eigenpair for $H^{\sim}(z)$. Moreover, $\lambda(-z)$ is a stable eigenvalue of $H^{\sim}(z)$, and $-\lambda(-z)$ is an unstable eigenvalue of $H(z)$.

We define a *stabilizing solution* to (5.2) as a solution $P(z)$, defined for all $z \in \mathfrak{A}(\alpha)$ for some $\alpha > 0$, such that $P^{\sim}(z) = P(z)$ and $A(z) - D(z)P(z)$ is stable for all $z \in \mathfrak{A}(\alpha)$. In other words, for every $z \in \mathfrak{A}(\alpha)$, $P(z)$ is $H(z)$-stabilizing in the sense of section 1 with the extra property $P^{\sim}(z) = P(z)$.

Using Theorem 2.1 we obtain sufficient conditions for the analyticity of $P(z)$ in some annulus around the unit circle $\mathbb{T}$.

THEOREM 5.2. *Suppose that for some $\alpha > 0$, $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$, and the following conditions hold:*

(1) *For every $z \in \mathbb{T}$, the matrix $H(z)$ has no eigenvalues on the imaginary axis.*
(2) *For every $z \in \mathbb{T}$, there exists an $H(z)$-stabilizing matrix.*
(3) *For at least one of the two points $z = \pm 1$, there exists a Hermitian $H(z)$-stabilizing matrix.*

Then for some β, $0 < \beta \leq \alpha$, we have that for every $z \in \mathfrak{A}(\beta)$, there exists a unique stabilizing solution $P(z)$ of (5.2), *and $P(z)$ is analytic in $\mathfrak{A}(\beta)$.*

Proof. By the continuity of eigenvalues, for some β', $0 < \beta' \leq \alpha$, $H(z)$ has no eigenvalues on the imaginary axis, for every $z \in \mathfrak{A}(\beta')$. By Theorem 2.1, there exists a meromorphic matrix function $P(z)$ on $\mathfrak{A}(\beta')$ such that $P(z)$ is the unique $H(z)$-stabilizing matrix for every $z \in \mathfrak{A}(\beta')$ not a pole of $P(z)$ and that there are no $H(z)$-stabilizing matrices if $z \in \mathfrak{A}(\beta')$ is a pole of $P(z)$. Condition (2) guarantees that the poles of $P(z)$ are outside $\mathbb{T}$. Thus, $P(z)$ is analytic in some annulus $\mathfrak{A}(\beta)$, $0 < \beta \leq \beta'$. It remains to prove that $P^{\sim}(z) = P(z)$. To this end, note that by (3), $P(1) = P(1)^*$ or $P(-1) = P(-1)^*$ holds. Suppose that $P(1) = P(1)^*$. (If $P(-1) = P(-1)^*$, the proof below works with obvious changes.) Then $P^{\sim}(1) = P(1)$, and all eigenvalues of $A(1) - D(1)P^{\sim}(1)$ are in the open left half plane. Therefore, for all z sufficiently close to 1, all eigenvalues of $A(z) - D(z)P^{\sim}(z)$ are also in the open left half plane. Thus $P^{\sim}(z)$ is $H(z)$-stabilizing for all such z. By the uniqueness of the $H(z)$-stabilizing matrix, we conclude that $P^{\sim}(z) = P(z)$ for all such z. But $P^{\sim}$ is analytic in $\mathfrak{A}(\beta)$ (because P is), and so we must have $P^{\sim}(z) = P(z)$ for all $z \in \mathfrak{A}(\beta)$. ∎

Theorems 5.2 and 4.1 provide the following useful corollary.

COROLLARY 5.3. *Suppose that for some $\alpha > 0$, $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$ and $(A(z), B(z), C(z))$ is stabilizable and detectable for all $z \in \mathbb{T}$. Then there exists β, $0 < \beta \leq \alpha$, such that for every $z \in \mathfrak{A}(\beta)$, there exists a unique stabilizing solution $P(z)$ of* (5.1) *and $P(z)$ is analytic in $\mathfrak{A}(\beta)$.*

The following theorem is another result asserting analyticity of stabilizing solutions, under suitable uniqueness hypothesis.

THEOREM 5.4. *Assume that for some $\alpha > 0$, $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$. If for every $z \in \mathfrak{A}(\alpha)$ there is a unique $H(z)$-stabilizing matrix $P(z)$ and at least one of the two matrices $P(1)$ and $P(-1)$ is Hermitian, then $P^{\sim}(z) = P(z)$ for all $z \in \mathfrak{A}(\alpha)$ (thus $P(z)$ is a stabilizing solution of* (5.2)*) and $P(z)$ is analytic in $\mathfrak{A}(\alpha)$.*

Proof. By Theorem 3.2, $P(z)$ is analytic in $\mathfrak{A}(\alpha)$. Taking the transformation $X(z) \mapsto X(\overline{1/z})^*$ in (5.1), we see that the graph subspace $G(P^{\sim}(z))$ is $H(z)$-invariant for all $z \in \mathfrak{A}(\alpha)$. Suppose that $P(1) = P(1)^*$. (If $P(-1) = P(-1)^*$, we use analogous considerations.) Then applying the arguments from the proof of Theorem 5.2 completes the proof. ∎

So to obtain more information about the size of the annulus of analyticity, we need to examine the existence of a stabilizing solution. An analogue of the following result is given in [2]. For completeness we include a slightly modified self-contained proof.

THEOREM 5.5. *Suppose that for some $\alpha > 0$, the following properties hold:*

(1) *Equation* (5.2) *has a stabilizing solution for all $z \in \mathbb{T}$.*

(2) *$H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$.*

(3) *$H(z)$ has no eigenvalues on the imaginary axis for all $z \in \mathfrak{A}(\alpha)$.*

(4) *The pair $(A(z), D(z))$ is stabilizable for all $z \in \mathfrak{A}(\alpha)$.*

(5) *$D(z)$ is in the factored form $D(z) = B(z)B^{\sim}(z)$, where $B(z)$ is $n \times m$ and analytic in $\mathfrak{A}(\alpha)$, and for every $z \in \mathfrak{A}(\alpha)$, if for some vectors x, y there holds $y^T D(z)x = 0$, then $y^T B(z) = 0$ or $B^{\sim}(z)x = 0$.*

Then for all $z \in \mathfrak{A}(\alpha)$, (5.2) *has a unique stabilizing solution $P(z)$.*

It is clear from the formulation of Theorem 5.5 that the hypothesis with regard to factorability of $D(z)$ as $B(z)B^{\sim}(z)$ is essential in the theorem; in particular, $D(z)$ is positive semidefinite for $|z| = 1$. Formally speaking, the proof goes through also in case $D(z) = B(z)D_0(z)B^{\sim}(z)$ for some $m \times m$ $D_0(z)$; however, property (5) is not satisfied unless D_0 is positive (or negative) definite.

Proof. (1) From item (1) we see that $H(z)$ has no eigenvalues for $z \in \mathbb{T}$ and its spectral subspace corresponding to the eigenvalues in the open left half plane is a graph subspace with dimension n. By continuity of eigenvalues, item (3) guarantees that this remains true for all $z \in \mathfrak{A}(\alpha)$.

(2) For $z \in \mathfrak{A}(\alpha)$, let $H(z)_-$ denote the restriction of $H(z)$ to the stable eigenspace of $H(z)$,

$$H(z)\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} H_-(z), \tag{5.5}$$

and the columns of $\left[\begin{smallmatrix} X_1(z) \\ X_2(z) \end{smallmatrix}\right]$ form a basis in the stable eigenspace. Thus

$$\begin{bmatrix} X_1^{\sim}(z) & X_2^{\sim}(z) \end{bmatrix} H^{\sim}(z) = H_-^{\sim}(z) \begin{bmatrix} X_1^{\sim}(z) & X_2^{\sim}(z) \end{bmatrix}, \tag{5.6}$$

and $H_-^{\sim}(z) := H_-(\overline{1/z})^*$ is a stable $n \times n$ matrix.

(3) We show that

$$X_1^{\sim}(z)X_2(z) = X_2^{\sim}(z)X_1(z). \tag{5.7}$$

Multiplying (5.5) from the left by $\begin{bmatrix} X_1^{\sim}(z) & X_2^{\sim}(z) \end{bmatrix} \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, we obtain

$$-\begin{bmatrix} X_1^{\sim}(z) & X_2^{\sim}(z) \end{bmatrix} H^{\sim}(z) \begin{bmatrix} X_2(z) \\ -X_1(z) \end{bmatrix} = (X_1^{\sim}(z)X_2(z) - X_2^{\sim}(z)X_1(z))H_-(z).$$

Using (5.6) we obtain

$$-H^{\sim}(z) \begin{bmatrix} X_1^{\sim}(z) & X_2^{\sim}(z) \end{bmatrix} \begin{bmatrix} X_2(z) \\ -X_1(z) \end{bmatrix} = (X_1^{\sim}(z)X_2(z) - X_2^{\sim}(z)X_1(z))H_-(z)$$

and the Sylvester equation

$$H_-^{\sim}(z)(X_1^{\sim}(z)X_2(z) - X_2^{\sim}(z)X_1(z)) + (X_1^{\sim}(z)X_2(z) - X_2^{\sim}(z)X_1(z))H_-(z) = 0.$$

Since both $H_-(z)$ and $H_-^{\sim}(z)$ are stable matrices, we have proven the claim (5.7).

We also claim that $X_1(z)$ is invertible if and only if $X_1^\sim(z)$ is. Suppose this is false, and $X_1(z)$ is invertible but $X_1^\sim(z)$ is not. Then there exists a $w \neq 0$ such that $w^* X_1^\sim(z) = 0$, and from (5.7) we have $w^* X_2^\sim(z) X_1(z) = 0$ and $w^* X_2^\sim(z) = 0$ which contradicts the fact that the columns of $\left[\begin{smallmatrix} X_1(z) \\ X_2(z) \end{smallmatrix}\right]$ form a basis (see Remark 5.1).

(4) We claim that if $X_1(z)$ is invertible, then $P(z) = X_2(z)X_1(z)^{-1}$ is the unique stabilizing solution. Multiplying (5.5) from the right by $X_1(z)$ gives

$$H(z)\begin{bmatrix} I \\ P(z) \end{bmatrix} = \begin{bmatrix} I \\ P(z) \end{bmatrix} X_1(z) H_-(z) X_1(z)^{-1}. \tag{5.8}$$

Now from (5.5) and (5.7), we see that multiplying (5.8) from the left by $\begin{bmatrix} P(z) & -I \end{bmatrix}$ gives zero on the right-hand side. Hence it yields the Riccati equation (5.1), and $P(z)$ is a solution. The symmetry property $P^\sim(z) = P(z)$ follows from (5.7). To show the stabilizing property, multiply (5.8) from the left by $[I\ 0]$ to obtain

$$\begin{bmatrix} A(z) & -D(z) \end{bmatrix} \begin{bmatrix} I \\ P(z) \end{bmatrix} = X_1(z) H_-(z) X_1(z)^{-1},$$

which reduces to

$$A(z) - D(z)P(z) = X_1(z)H_-(z)X_1(z)^{-1}.$$

So $A(z) - D(z)P(z)$ is similar to the stable matrix $H_-(z)$. Note that

$$A^\sim(z) - P^\sim(z)D(z) = A^\sim(z) - P(z)D(z)$$

is similar to the stable matrix $H_-^\sim(z)$, and so it is also stable.

For the uniqueness, we suppose that we have two stabilizing solutions $P_1(z)$ and $P_2(z)$. Then $X(z) := P_1(z) - P_2(z)$ satisfies the Sylvester equation

$$(A^\sim(z) - P_1(z)D(z))X(z) + X(z)(A(z) - D(z)P_2(z)) = 0.$$

Since both coefficient matrices are stable, the only solution is $X(z) = 0$.

(5) It remains to show that $X_1(z)$ is invertible for all $z \in \mathfrak{A}(\alpha)$. Multiplying (5.5) from the left by $\left[\ X_2^\sim(z) \quad -X_1^\sim(z)\ \right]$ and using (5.7), we obtain

$$\begin{bmatrix} X_2^\sim(z) & -X_1^\sim(z) \end{bmatrix} H(z) \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = 0,$$

which is equivalent to

$$X_1^\sim(z)A^\sim(z)X_2(z) + X_2^\sim(z)A(z)X_1(z) + X_1^\sim(z)Q(z)X_1(z) = X_2^\sim(z)D(z)X_2(z). \tag{5.9}$$

Suppose that there exists $v_0 \neq 0$ such that $X_1(z_0)v_0 = 0$ for some $z_0 \in \mathfrak{A}(\alpha)$. Then from part (1) of this proof there exists $w_0 \neq 0$ such that $w_0^* X_1^\sim(z_0) = 0$ or equivalently $X_1(\overline{1/z_0})w_0 = 0$. Hence multiplying (5.9) from the left by w_0^* and the right by v_0, we obtain $w_0^* X_2^\sim(z_0)D(z_0)X_2(z_0)v_0 = 0$. But item (5) implies that we must have

$$\text{(a)} \quad B^\sim(z_0)X_2(z_0)v_0 = 0 \qquad \text{or} \qquad \text{(b)} \quad w_0^* X_2^\sim(z_0)B(z_0) = 0.$$

Suppose that (a) holds. Then (5.5) implies that $X_1(z_0)H_-(z_0)v_0 = 0$, or, in other words, $\ker X_1(z_0)$ is $H_-(z_0)$-invariant. Noting that (b) is equivalent to

$$B^\sim(1/z_0)X_2(1/z_0)w_0 = 0,$$

we see that (b) implies that $\ker X_1(1/z_0)$ is $H_-(1/z_0)$ -invariant. So either $\ker X_1(z_0)$ is $H_-(z_0)$-invariant or $\ker X_1(1/z_0)$ is $H_-(1/z_0)$-invariant. Assuming that the first holds, there exists $v \in \ker X_1(z_0) \setminus \{0\}$ and λ with $\operatorname{Re}\lambda < 0$ such that $H_-(z_0)v = \lambda v$. Multiplying (5.5) (with $z = z_0$) by v, we obtain

$$\begin{bmatrix} -D(z_0) \\ -A^{\sim}(z_0) \end{bmatrix} X_2(z_0)v = \begin{bmatrix} 0 \\ \lambda X_2(z_0)v \end{bmatrix}.$$

Hence

$$D(z_0)X_2(z_0)v = 0, \quad (A^{\sim}(z_0) + \lambda I)X_2(z_0)v = 0. \tag{5.10}$$

Note that because the columns of $\left[\begin{smallmatrix} X_1(z_0) \\ X_2(z_0) \end{smallmatrix}\right]$ are linearly independent, we must have $X_2(z_0)v \neq 0$. Now in view of (5.10), $(A^{\sim}(z_0), D(z_0)) = (A(\overline{1/z_0})^*, D(z_0))$ is not detectable at z_0. This implies that $(A(\overline{1/z_0}), D(z_0)^*) = (A(\overline{1/z_0}), D(\overline{1/z_0}))$ is not stabilizable at $\overline{1/z_0}$. Since $\overline{1/z_0} \in \mathfrak{A}(\alpha)$, this contradicts the stabilizability assumption. If $\ker X_1(1/z_0)$ is $H_-(1/z_0)$-invariant, the argument is analogous. □

Combining Theorem 5.5 with Theorem 5.4 gives our main result on the analyticity of solutions to (5.1).

Corollary 5.6. *Suppose that*

(1) *$(A(z), B(z), C(z))$ is stabilizable and detectable for every $z \in \mathbb{T}$;*
(2) *for some $\alpha > 0$, the matrix function $H(z)$ is analytic in the annulus $\mathfrak{A}(\alpha)$;*
(3) *the matrix $H(z)$ has no eigenvalues on the imaginary axis for all $z \in \mathfrak{A}(\alpha)$;*
(4) *$(A(z), B(z)B^{\sim}(z))$ is stabilizable for all $z \in \mathfrak{A}(\alpha)$;*
(5) *for every $z \in \mathfrak{A}(\alpha)$, if for some vectors x, y there holds $y^T B(z)B^{\sim}(z)x = 0$, then $y^T B(z) = 0$ or $B^{\sim}(z)x = 0$.*

Then for all $z \in \mathfrak{A}(\alpha)$, (5.1) has a unique stabilizing solution $P(z)$ that is analytic on $\mathfrak{A}(\alpha)$.

We remark that the conditions in items (1)–(4) of Theorem 5.5 are necessary (see Theorem 2.1), but the condition in item (5) of Theorem 5.5 is very restrictive. However, it does hold for rank one input operators of the form $B(z) = b_1(z)b_2^{\sim}(z)$, where $b_1(z) \in \mathbb{C}^n$, $b_2(z) \in \mathbb{C}^m$, provided that the scalar analytic function $b_2^{\sim}(z)b_2(z)$ has no zeros in the annulus $\mathfrak{A}(\alpha)$. This case occurs frequently in applications, but it is easy to construct examples for which the condition in item (5) of Theorem 5.5 is not necessary to achieve analyticity.

Example 5.7. Consider the class of systems

$$A(z) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C(z) = \begin{bmatrix} c(z) & 0 \\ 0 & c(z) \end{bmatrix},$$

and assume that $c(z) \neq 0$, $z \in \mathbb{T}$, and $q(z) := c^{\sim}(z)c(z)$ is analytic in some annulus $\mathfrak{A}(\alpha)$. Then $H(z)$ will have imaginary eigenvalues only if the following equation has a real solution ω:

$$\omega^4 + 2q(z)\omega^2 + q(z) + q(z)^2 = 0.$$

So we choose $c(z)$ such that this equation has no real solutions in $\mathfrak{A}(\alpha)$. Then all the conditions in items (1)–(4) of Theorem 5.5 are satisfied, but the condition in item (5) is not satisfied.

The solution to the Riccati equation (5.1) is

$$
\begin{aligned}
P(z) &= \begin{bmatrix} \sqrt{-2q^2+2q\sqrt{q^2+q}} & -q+\sqrt{q^2+q} \\ -q+\sqrt{q^2+q} & \sqrt{-2q(1+q)+2(1+q)\sqrt{q^2+q}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2qp_3} & p_3 \\ p_3 & \sqrt{2(1+q)p_3} \end{bmatrix}, \quad p_3 = -q+\sqrt{q^2+q}, \quad q = q(z) = c^{\sim}(z)c(z).
\end{aligned}
$$

Under the above assumptions, $P(z)$ will be the unique stabilizing solution in the annulus $\mathfrak{A}(\alpha)$.

It is possible to generate infinitely many examples of this class, but we give two. If $c(z) = z$, then $q(z) = 1$ and the unique stabilizing solution is

$$
P = \begin{bmatrix} \sqrt{2}t & t^2 \\ t^2 & 2t \end{bmatrix}, \quad \text{where} \quad t = \sqrt{\sqrt{2}-1}.
$$

If $c(z) = 2+z$, then $q(z) = (2+\frac{1}{z})(2+z)$, the function $H(z)$ is analytic in $\mathfrak{A}(\log 2) = \{z : 1/2 < |z| < 2\}$, and $(A(z), B(z), C(z))$ is stabilizable and detectable in $\mathfrak{A}(\log 2)$. Computations show that $H(z)$ has no imaginary eigenvalues for every $z \in \mathfrak{A}(\log 2)$, and so $P(z)$ is the unique stabilizing solution for all $z \in \{z : 1/2 < |z| < 2\}$.

6. Fourier transforms of systems arising from PDEs.

A classic method of analyzing PDEs is to take Fourier transforms (see Hörmander [15]). This results in matrix multiplication operators on suitable Banach function spaces, for example, $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)$, where $\mathbb{J}$ denotes the imaginary axis (see [14]). An $n \times n$ complex matrix $A(\cdot)$ induces a *matrix multiplication operator* M_A defined by

$$
\begin{aligned}
&D(M_A) := \{f \in \mathbf{L}_2(\mathbb{J}; \mathbb{C}^n) : A(\lambda)f(\lambda) \in \mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)\}, \\
&M_A f(\lambda) = A(\lambda)f(\lambda), \quad \text{for all } f \in D(M_A), \quad \text{for all } \lambda \in \mathbb{J}.
\end{aligned}
$$

As explained in [2], after taking Fourier transforms a controlled partial differential system takes the form (4.4) with the state space $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^n)$, input space $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^m)$, and output space $\mathbf{L}_2(\mathbb{J}; \mathbb{C}^p)$.

If $A(\lambda), B(\lambda), C(\lambda)$ are continuous for $\lambda \in \mathbb{J}$, it can also be written in the form (4.5) for all z replaced by $\lambda \in \mathbb{J}$. So this yields another class of spatially invariant systems as defined in [2] with the Riccati equation

$$
A(\lambda)^* P(\lambda) + P(\lambda)A(\lambda) - P(\lambda)B(\lambda)B(\lambda)^* P(\lambda) + C(\lambda)^* C(\lambda) = 0.
$$

We remark that since our matrices are continuous, $P(\lambda)$ will also be continuous (see [17, Theorem 11.2.1]).

The situation is analogous to that for platoon-type systems with the important difference that the platoon-type systems are parametrized over the unit circle, whereas the PDE-type systems are parametrized over the imaginary axis which is not compact. Analogously to the platoon case, for an implementable control law, the feedback gain should be localized. Typically the control law will have the form

$$
u(x,t) = \int_{-\infty}^{\infty} \kappa(x-\xi) z(\xi,t)\, \mathrm{d}\xi,
$$

where $z(x,t)$ is the solution of the controlled PDE, and κ is the distribution which is the inverse Fourier transform of $-B(s)^*P(s)$. Suppose that κ is a continuous function. Then this control law can be approximated by a localized control law provided that the kernel decays exponentially fast to zero as $|x| \to \infty$. A first step in proving this exponential decaying property is to establish the extension of $P(\lambda)$ to a function of $P(s)$ that is analytic in a vertical strip around the imaginary axis. This was done in [2, Theorem 6] for the special case that $A(s), B(s), C(s)$ are all rational analytic functions in a strip around the imaginary axis (and certain extra assumptions). The continuity holds for λ on the imaginary axis (under the extra assumptions); see [24, 17] and the references therein, and for the case of real coefficients in a more general setting, see [8].

In the next section we give several new results on the analyticity for the general nonrational case.

7. Analyticity of Riccati equation solutions on a vertical strip. Following the line in section 5, we seek solutions to the following nonstandard Riccati equation which are analytic in a strip around the imaginary axis (cf. section 6):

$$A^{\sim}(s)P(s) + P(s)A(s) - P(s)B(s)B^{\sim}(s)P(s) + C^{\sim}(s)C(s) = 0, \tag{7.1}$$

where $A(s)$, $B(s)$, $C(s)$ are matrix valued functions of sizes $n \times n$, $n \times p$, and $q \times n$, respectively, of the variable s in the strip $\mathfrak{S}(\alpha) := \{s \in \mathbb{C} : |\mathrm{Re}\,(s)| < \alpha\}$, for some $\alpha > 0$, but in contrast with section 5, we now have $A^{\sim}(s) := A(-\overline{s})^*$. As in section 5, we consider also a more general form of (7.1), where the coefficients of quadratic and free terms are not factored:

$$\begin{aligned} &A^{\sim}(s)P(s) + P(s)A(s) - P(s)D(s)P(s) + Q(s) = 0, \\ &D^{\sim}(s) = D(s), \quad Q^{\sim}(s) = Q(s) \quad \text{for all } s \in \mathfrak{S}(\alpha). \end{aligned} \tag{7.2}$$

The Hamiltonian matrix function $H(s)$ is defined by the same formula (5.3). Furthermore, formula (5.4) and Remark 5.1 remain valid.

We define a *stabilizing solution* to (7.2) to be a solution $P(s)$ such that $P^{\sim}(s) = P(s)$ and the matrix $A(s) - D(s)P(s)$ is stable for all $s \in \mathfrak{S}(\alpha)$. The following theorems are the analogues of Theorems 5.2, 5.4, and 5.5 in section 5.

THEOREM 7.1. *Assume that for some $\alpha > 0$, $H(s)$ is analytic in the strip $\mathfrak{S}(\alpha)$. If for every $s \in \mathfrak{S}(\alpha)$ there is a unique $H(s)$-stabilizing matrix $P(s)$ and the matrix $P(0)$ is Hermitian, then $P^{\sim}(s) = P(s)$ for all $s \in \mathfrak{S}(\alpha)$ (thus $P(s)$ is a stabilizing solution of* (7.2)*) and $P(s)$ is analytic in $\mathfrak{S}(\alpha)$.*

The proof is analogous to that of Theorem 5.4.

THEOREM 7.2. *Suppose that $H(s)$ is a rational matrix function and that the following conditions hold:*

(1) *There is $\alpha > 0$ such that $H(s)$ is analytic in the strip $\mathfrak{S}(\alpha)$ and for every $s \in \mathfrak{S}(\alpha)$ the matrix $H(s)$ has no eigenvalues on the imaginary axis.*
(2) *For every $z \in \mathbb{J}$, there exists an $H(s)$-stabilizing matrix.*
(3) *There exists a Hermitian $H(0)$-stabilizing matrix.*

Then there exists β, $0 < \beta \le \alpha$, such that for every s in the strip $\mathfrak{S}(\beta)$, there exists a unique stabilizing solution $P(s)$ of (7.2)*, and $P(s)$ is analytic in $\mathfrak{S}(\beta)$.*

Proof. By Theorem 2.1, there exists a meromorphic matrix function $P(s)$ on $\mathfrak{S}(\alpha)$ such that $P(s)$ is a unique $H(s)$-stabilizing matrix for every $s \in \mathfrak{S}(\alpha)$ that is not a pole of $P(s)$, and there are no $H(s)$-stabilizing matrices if $s \in \mathfrak{S}(\alpha)$ is a pole of $P(s)$.

Condition (2) guarantees that $P(s)$ has no poles on $\mathbb{J}$, and Theorem 2.4 guarantees that the number of poles of $P(s)$ in $\mathfrak{S}(\alpha)$ is finite. Thus, $P(s)$ is analytic in some strip $\mathfrak{S}(\beta)$, $0 < \beta \leq \alpha$. It remains to prove that $P^{\sim}(s) = P(s)$. To this end, note that by (3), $P(0) = P(0)^*$ holds, and repeat the arguments from the proof of Theorem 5.2. $\square$

Note that $H(s)$ is allowed to have a pole at infinity under the hypotheses of Theorem 7.2. We also point out that the hypothesis of $H(s)$ being rational yields at most finitely many poles of $P(s)$ (Theorem 2.4), and as a result, the conclusions of the theorem hold for some strip $\mathfrak{S}(\beta)$. Assuming that $H(s)$ is merely analytic in $\mathfrak{S}(\alpha)$, the conclusions of the theorem would hold for some open set $\Omega \subseteq \mathfrak{S}(\beta)$ such that $z \in \Omega \ \Rightarrow \ -z \in \Omega$, but Ω need not a be a strip of the form $\mathfrak{S}(\beta)$. The difference with the analogous result in Theorem 5.4 lies in the fact that, unlike $\mathfrak{A}(\alpha)$, $\mathfrak{S}(\alpha)$ is not compact. The translation of Theorem 5.5 to the strip yields the following existence result that was proven in [2, Theorem 6].

THEOREM 7.3. *Suppose that*

(1) *Equation* (7.2) *has a stabilizing solution for all* $s \in \mathbb{J}$*;*
(2) *for some* $\alpha > 0$*,* $H(s)$ *is analytic in the strip* $\mathfrak{S}(\alpha)$*;*
(3) $H(s)$ *has no eigenvalues on the imaginary axis for all* $s \in \mathfrak{S}(\alpha)$*;*
(4) $(A(s), D(s))$ *is stabilizable for all* $s \in \mathfrak{S}(\alpha)$*;*
(5) $D(s)$ *is in the factored form* $D(s) = B(s)B^{\sim}(s)$*, where* $B(s)$ *is* $n \times m$ *and analytic in* $\mathfrak{S}(\alpha)$*, and for every* $s \in \mathfrak{S}(\alpha)$*, if for some vectors* x, y *there holds* $y^T D(s)x = 0$*, then* $y^T B(s) = 0$ *or* $B^{\sim}(s)x = 0$*.*

Then for all $s \in \mathfrak{S}(\alpha)$*,* (7.2) *has a unique stabilizing solution* $P(s)$*.*

Combining Theorems 7.3 and 7.1 yields the proof of the following result which generalizes that in [2, Theorem 6], where it was assumed that $H(s)$ was rational.

COROLLARY 7.4. *Suppose that* $A(s), B(s), C(s)$ *are matrix functions of suitable sizes, and the following conditions hold:*

(1) $(A(s), B(s), C(s))$ *is stabilizable and detectable for all* $s \in \mathbb{J}$*.*
(2) *For some* $\alpha > 0$*,* $A(s), B(s), C(s)$ *are analytic in the strip* $\mathfrak{S}(\alpha)$*.*
(3) $H(s)$ *has no eigenvalues on the imaginary axis for all* $s \in \mathfrak{S}(\alpha)$*.*
(4) $(A(s), B(s)B^{\sim}(s))$ *is stabilizable for all* $s \in \mathfrak{S}(\alpha)$*.*
(5) *For every* $s \in \mathfrak{S}(\alpha)$*, if for some vectors* x, y *there holds* $y^T B(s)B^{\sim}(s)x = 0$*, then* $y^T B(s) = 0$ *or* $B^{\sim}(s)x = 0$*.*

Then for all $s \in \mathfrak{S}(\alpha)$*,* (7.1) *has a unique stabilizing solution* $P(s)$ *that is analytic on* $\mathfrak{S}(\alpha)$*.*

This corollary explains the difference between Examples 2.2 and 2.3. In both examples, $H(s)$ is analytic in $\mathbb{C}$, and $H(s)$ has no eigenvalues on the imaginary axis for $s \in \mathfrak{S}(\sqrt{2/3})$. In Example 2.2 $(A(s), B(s)) = (1 + s, 1 + \sqrt{2}s)$ is stabilizable in $\mathfrak{S}(1/\sqrt{2})$, and so by Corollary 7.4, $P(s)$ is the stabilizing solution in $\mathfrak{S}(1/\sqrt{2})$. In Example 2.3, $(A(s), B(s)) = (-1 + s, 1 + \sqrt{2}s)$ is stabilizable in $\mathbb{C}$, and so $P(s)$ is the stabilizing solution in the larger strip $\mathfrak{S}(\sqrt{2/3})$.

8. Remark. In this paper we have considered the particular Riccati equations that occur most frequently in control applications. We note that Riccati equations with symmetries of the types $A^{\sim}(z) := A(z^{-1})^T$ (rather than $A^{\sim}(z) := A(\overline{z^{-1}})^*$) and $A^{\sim}(s) := A(-s)^T$ (rather than $A^{\sim}(s) := A(\overline{-s})^*$) can be treated in an analogous manner with analogous results.

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